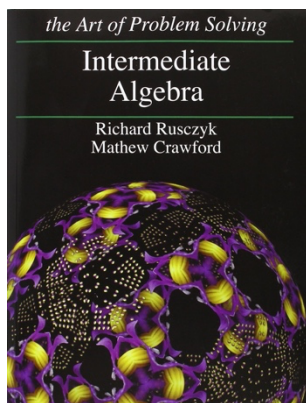


# Algebra II (AoPS) – Class Description



**Textbook and Workbook:** Intermediate Algebra, 2<sup>nd</sup> Edition, by Richard Rusczyk, the Art of Problem Solving series.

## **Prerequisite**

This class is typically for ninth graders or above, or advanced eighth graders with solid foundations in Algebra. Students should have completed the course of Algebra I, and preferably have completed the course of Geometry. Students must be proficient in fundamentals of Algebra, such as basic concepts and problem solving skills in the topics of basic rules of exponents and radicals, ratio, rate, percent, quadratic factorization, solving and graphing linear and quadratic equations, complex numbers, solving and graphing linear inequalities, fundamentals of linear, quadratic and rational functions, operation of functions, inverse functions, polynomial arithmetic, etc.

## **The Course**

- 1) Topics covered in this course include a quick review of basic algebra topics, followed by mid to advance topics of complex numbers, quadratics and conic sections, analytic geometry, polynomials, multivariable expressions, sequences and series, identities, inequalities, exponents and logarithms, polynomial and rational functions, exponential and logarithmic functions, piecewise-defined functions, functional equations, and much more;
- 2) The course is structured to inspire the students to explore and develop new ideas. Each class starts with brief introduction of core concepts and examples, followed by challenging practice problems, so the students have a chance to solve them under the guidance of the teacher. Step by step solutions to these practice problems are explained in detail, through which new techniques are taught. Important facts and powerful problem solving approaches are highlighted throughout the course. Homework and exams (if needed) are used to evaluate the level of mastering.

## **Teacher's Focus:**

- Introduce, explain, and apply core algebraic concepts with systemic and comprehensive class notes and examples;
- Emphasize the application of core concepts, and problem solving strategies and skills, through challenging problems with step by step solution;
- Foster students' interest and confidence by guided practice and class discussion.

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# Joe Zhou



## **Subjects taught**

Algebra 1 (AoPS), Geometry (AoPS), Algebra 2 (AoPS)

## **Experience**

I have been teaching Algebra 1 (AoPS), Geometry (AoPS), and Algebra 2 (AoPS) courses at Hope Chinese School for over three years.

I have also been working at C2 Education as math tutor for two and a half years, including subjects of Pre-algebra, Algebra 1 and 2, Geometry, Pre-calculus, SAT and ACT Math.

## **Education**

MBA (Kellogg School of Management)

MS Computer Science (University of Illinois at Chicago)

BA Biomedical Engineering (Tsinghua University)

My passion is helping students achieve their full potential.



**Are You Ready?***Intermediate Algebra*, by R. Rusczyk and M. Crawford

If you've mastered working with proportions, multiplying simple algebraic expressions, the basics of logarithms, simple functions and basic linear equations as illustrated in the problems below, then you are ready for the Art of Problem Solving textbook, **Intermediate Algebra**. (Answers to these problems are on the following page.)

**1. Algebraic Expressions** Simplify each of the following expressions.

- (a)  $3x + 8x$
- (b)  $(4y - 7) - 3(10y - 39)$
- (c)  $(2x + 1)(3x + 4)$
- (d)  $(x + y)^3$

**2. Linear Equations** Solve each of the following linear equations.

- (a)  $x + 12 = 20$
- (b)  $3x = 36$
- (c)  $4y + 9 = -17$
- (d)  $-14x + 47 = 8x - 84$

**3. Factoring** Factor each of the following expressions.

- (a)  $x^2 - 7x + 10$
- (b)  $x^2y + 5xy - 36y$
- (c)  $x^3 - 6x^2 + 11x - 6$
- (d)  $x^3 - x^2 + x - 1$

**4. Means, Medians, and Modes**

- (a) What are the mean, median, and mode of the numbers 4, 5, 7, 7, and 7?
- (b) If the sum of 10 numbers is 90, what is the mean of the 10 numbers?
- (c) Jen scored an 88 on her midterm. Out of the 14 other students in her class, 7 of them scored lower than she did. What was the median of the class midterm scores?

**5. Proportions** Evaluate each of the following problems involving proportionality.

- (a) If  $x$  and  $y$  are directly proportional and  $x = 5$  when  $y = 30$ , what is  $y$  when  $x$  is 30?
- (b) If  $x$  and  $y$  are inversely proportional, what happens to  $x$  when  $y$  is increased by 25%?
- (c) If  $x$  and  $y^2$  are inversely proportional and  $x = 4$  when  $y = 4$ , what is the value of  $x$  when  $y = \frac{1}{2}$ ?

**6. Logarithms** Express each of the following as a single number.

- (a)  $\log_2 1$
- (b)  $\log_3 \frac{1}{81}$
- (c)  $\log_3 3^7$

**7. Quadratic Equations** Solve each of the following quadratic equations:

(a)  $x^2 - 5x + 6 = 0$

(b)  $x^2 + 2x - 24 = 0$

(c)  $6x^2 - 13x + 6 = 0$

(d)  $x^2 + 6x + 13 = 0$

(e)  $x^2 + x + 1 = 0$

**8. Functions**

(a) If  $f(x) = x^2 + 6x + 18$ , for what value(s) of  $x$  does  $f(x) = 9$ ?

(b) If  $f(x) = 2f(x - 1)$  for all integers  $x$ , and  $f(n) = 3$  for some integer  $n$ , find the value of  $[f(n - 5)][f(n + 5)]$ .

(c) If  $f(x) = 3x(2^x)$ , how many times does the graph of  $f(x)$  cross the  $x$ -axis?

(d) If  $f(x) + f(2 - x) = 4$  for all  $x$ , find  $f(y - 2) + f(4 - y)$

In addition to these kinds of problems, students should be able to construct graphs on the coordinate plane such as  $y = 3x + \frac{7}{3}$ ,  $4x + 5y = 7$ , and  $y = |x + 2|$ .

**Don't look at the next page until you've attempted all the problems!**